2007 Form B

$f(x) = 6e^{-x/3}$ $\qquad$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

a) $f(x) = 6\left[1 + (-x/3) + \frac{(-x/3)^2}{2!} + \frac{(-x/3)^3}{3!} + \dots + \frac{(-x/3)^n}{n!}\right]$

$$\boxed{= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \dots + \frac{(-1)^n \cdot 6x^n}{3^n n!}}$$

b) $g(x) = \int_0^x f(t)\,dt = \int_0^x \left(6 - 2t + \frac{t^2}{3} - \frac{t^3}{27} + \dots\right) dt$

$$= 6t - t^2 + \frac{t^3}{9} - \frac{t^4}{108} + \dots + \frac{(-1)^n \cdot 6x^{n+1}}{3^n n!(n+1)} \quad n!(n+1) = (n+1)!$$

$$\boxed{= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{108} + \dots + \frac{(-1)^n \cdot 6x^{n+1}}{3^n \cdot (n+1)!}}$$

c) $h(x) = k f'(ax)$ $\qquad$ $h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} = e^x$

$\downarrow$

$e^x = k\left(-2e^{-ax/3}\right)$

$e^x = -2ke^{-ax/3}$

$f(x) = 6e^{-x/3}$ $\qquad$ $f'(x) = 6 \cdot \frac{-1}{3} e^{-x/3}$

$= -2e^{-x/3}$

$f'(ax) = -2e^{-ax/3}$

$1 = -2k$ $\qquad$ $1 = -a/3$

$\boxed{k = -1/2}$ $\qquad$ $3 = -a$

$\boxed{a = -3}$

2008

a) Using the table: $\boxed{T(x) = 80 + 128(x-2)}$

$\boxed{T(1.9) = 67.2}$

$\boxed{\text{since } h''(2) = \frac{488}{3} \rightarrow \text{cc up} \therefore T(1.9) > h(1.9)}$

b) $T(x) = 80 + 128(x-2) + \frac{488/3\,(x-2)^2}{2!} + \frac{448/3\,(x-2)^3}{3!}$

$$\boxed{T(x) = 80 + 128(x-2) + \frac{488(x-2)^2}{6} + \frac{448(x-2)^3}{18}}$$

$\boxed{T(1.9) = 67.988}$

c) on $[1,3] \rightarrow h^4(x) = \frac{584}{9}$

Next term

$$\text{error} = \frac{h^4(2)(x-2)^4}{4!} < 3\times10^{-4}$$

$$\frac{\frac{584}{9}(1.9-2)^4}{4!} < 3\times10^{-4}$$

$$2.704\times10^{-4} < 3\times10^{-4} \checkmark$$

2008 Form B

$$f(x) = \frac{2x}{1+x^2} \longrightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots + (-1)^n x^n$$

a) $$\frac{2x}{1+x^2} = 2x\left[1 - (x^2) + (x^2)^2 - (x^2)^3 + \dots + (-1)^n (x^2)^n\right]$$

$$\boxed{= 2x - 2x^3 + 2x^5 - 2x^7 + \dots + (-1)^n 2x^{2n+1}}$$

b) No. The series goes to $\overset{\text{two or}}{\text{zero}}$ when evaluated @ $x=1$.

ex: $2(1) - 2(1)^3 + 2(1)^5 - 2(1)^7 + \dots$

c) $$\ln(1+x^2) = \int_0^x \frac{2t}{1+t^2}\,dt = \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \dots\right) dt$$

$$= t^2 - \frac{t^4}{2} + \frac{t^6}{3} - \frac{t^8}{4} + \dots \Big|_0^x$$

$$\boxed{= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots}$$

d) $$\Big|\underset{\text{est}}{A} - \underset{\text{actual}}{\ln(\overset{x=1/2}{5/4})}\Big| < \frac{1}{100}$$

$$\ln(5/4) = (1/2)^2 - \frac{(1/2)^4}{2} + \frac{(1/2)^6}{3} - \frac{(1/2)^8}{4} + \dots$$

Partial Sums

$S_1 = 1/4$

$S_2 = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$

$S_3 = \frac{7}{32} - \frac{1}{192} = \frac{41}{192}$ ∴ The 3rd term will make the error $< \frac{1}{100}$

$$\left|\frac{41}{192} - \ln(5/4)\right| < \frac{1}{100}$$

$$.0096 < .01$$

error → next term: $\frac{x^6}{3}$

@ $x = 1/2$ $\quad \frac{(1/2)^6}{3} < \frac{1}{100}$

$\frac{1}{192} < \frac{1}{100}$ ✓

2009

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \qquad f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2} \quad ; \quad f(1) = 1$$

a) $$e^{(x-1)^2} = 1 + (x-1)^2 + \frac{[(x-1)^2]^2}{2!} + \frac{[(x-1)^2]^3}{3!} + \dots + \frac{[(x-1)^2]^n}{n!}$$

$$\boxed{= 1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \dots + \frac{(x-1)^{2n}}{n!}}$$

b) $$f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2} = \frac{\left[\cancel{1} + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \dots + \frac{(x-1)^{2n}}{n!}\right] \cancel{-1}}{(x-1)^2}$$

$$\boxed{= 1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{3!} + \frac{(x-1)^6}{4!} + \dots + \frac{(x-1)^{2n}}{(n+1)!}}$$

c) RATIO:

$$\lim_{n\to\infty}\left[\frac{\overset{\cancel{(x-1)^{2n}}\,(x-1)^2}{(x-1)^{2(n+1)}}}{\underset{(n+2)\cancel{(n+1)!}}{(n+2)!}} \cdot \frac{\cancel{(n+1)!}}{\cancel{(x-1)^{2n}}}\right] = \cancel{\lim_{n\to\infty}} \overset{\nearrow 0}{\frac{1}{n+2}} \cdot (x-1)^2 = 0(x-1)^2 = 0 < 1$$

Always

$$\boxed{\therefore -\infty < x < \infty}$$

d) $$f'(x) = \frac{2(x-1)^1}{2!} + \frac{4(x-1)^3}{3!} + \frac{6(x-1)^5}{4!} + \dots + \frac{2n(x-1)^{2n-1}}{(n+1)!}$$

$$f''(x) = \frac{2!(x-1)^0}{2!} + \frac{4\cdot3(x-1)^2}{3!} + \frac{6\cdot5(x-1)^4}{4!} + \dots + \frac{2n(2n-1)(x-1)^{2n-2}}{(n+1)!}$$

$\boxed{f''(x) > 0 \text{ for every } x \quad \therefore f(x) \text{ has no pts. of inflection}}$

2009 Form B

$$f(x) = 1 + (x+1) + (x+1)^2 + \ldots + (x+1)^n + \ldots$$

a) RATIO:

$$\lim_{n\to\infty} \frac{(x+1)^{n+1}}{(x+1)^n} = x+1 \qquad (x+1)^n \cdot (x+1)^1$$

$|x+1| < 1$

$-1 < x+1 < 1$

$-2 < x < 0$

check the end-pts:

$x = -2 \to \sum_{n=0}^{\infty} (-1)^n \to$ div

$x = 0 \to \sum_{n=0}^{\infty} (1)^n \to$ div.

$\boxed{\text{so... } (-2, 0)}$

b) $\text{Sum} = \frac{a_1}{1-r} = \frac{1}{1-(x+1)} = \boxed{\frac{-1}{x}}$

c) $g(x) = \int_{-1}^{x} f(t)\,dt \qquad g(-\tfrac{1}{2}) = \int_{-1}^{-1/2} -\tfrac{1}{x}\,dx$

$$= -\ln|x| \Big|_{-1}^{-1/2}$$

$$= -\ln|-\tfrac{1}{2}| + \cancel{\ln|-1|}^{\,0}$$

$$= -\ln(\tfrac{1}{2}) \to \ln(\tfrac{1}{2})^{-1} \to \boxed{\ln(2)}$$

d) $h(x) = f(x^2-1) = 1 + (x^2 \cancel{-1} \cancel{+1}) + (x^2)^2 + (x^2)^3 + \ldots + (x^2)^n$

$$\boxed{= 1 + x^2 + x^4 + x^6 + \ldots + x^{2n} + \ldots}$$

$$h(\tfrac{1}{2}) = \underbrace{1 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^4 + \ldots + (\tfrac{1}{2})^{2n}}_{\text{Sum} = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \boxed{4/3}} \to (\tfrac{1}{4})^n$$

2010

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases} \qquad g(x) = 1 + \int_0^x f(t)\,dt$$

a) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots + \dfrac{(-1)^n x^{2n}}{(2n)!}$

$$f(x) = \frac{\left[\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right] - \cancel{1}}{x^2} = \boxed{\frac{-1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \frac{x^6}{8!} - \ldots + \frac{(-1)^{n+1} x^{2n}}{(2n+2)!}}$$

b) $f'(0) = 0 \rightarrow x = 0$ is a crt. pt.

$f''(0) = \frac{1}{12} \rightarrow$ cc up $\boxed{\therefore x = 0 \text{ must be a minimum}}$

$\dfrac{1}{4!} = \dfrac{f''(0)}{2!}$

$\dfrac{2!}{4!} = f''(0)$

$4 \cdot 3 \cdot \cancel{2!}$

c) $g(x) = 1 + \int_0^x f(t)\,dt = 1 + \int_0^x \left(-\frac{1}{2} + \frac{t^2}{4!} - \frac{t^4}{6!} + \ldots\right) dt$

$= 1 + \dfrac{-t}{2} + \dfrac{t^3}{3 \cdot 4!} - \dfrac{t^5}{5 \cdot 6!} + \ldots \Big|_0^x$

$\boxed{= 1 - \dfrac{x}{2} + \dfrac{x^3}{3 \cdot 4!} - \dfrac{x^5}{5 \cdot 6!}}$

use a 3rd degree to estimate

d) $g(1) \approx 1 - \frac{1}{2} + \frac{1}{3 \cdot 4!} - \cancel{\frac{1}{5 \cdot 6!}} \boxed{= \dfrac{37}{42}}$

error $\rightarrow$ next term : $\cancel{\dfrac{x^7}{7 \cdot 8!}}$ $\dfrac{x^5}{5 \cdot 6!}$

@ $x = 1$ $\dfrac{(1)^5}{5 \cdot 6!} < \dfrac{1}{6!}$

$\dfrac{1}{3600} < \dfrac{1}{720}$ ✓