Series Packet - Day 1: 1979-1990

1979 BC4:

$f(x) = \frac{1}{1-2x}$

a) $a_0 = 1,\ r = 2x \rightarrow \boxed{1 + 2x + 4x^2 + 8x^3 + \ldots + (2x)^n}$

b) $\sum_{n=0}^{\infty} (2x)^n \rightarrow r = 2x \rightarrow |2x| < 1 \rightarrow |x| < 1/2$: Cen: $x=0$, R: $1/2$

IOC: $\boxed{(-1/2, 1/2)}$

$x = -1/2$: $(2 \cdot -\frac{1}{2})^n = (-1)^n \rightarrow r = -1 \rightarrow$ Div.

$x = 1/2$: $(2 \cdot \frac{1}{2})^n = (1)^n \rightarrow r = 1 \rightarrow$ Div.

c) $f(-1/4) = \frac{1}{1-2(-1/4)} = \frac{1}{1+1/2} = \frac{1}{3/2} = \frac{2}{3}$ = actual value

1% of $\frac{2}{3} = \frac{1}{100} \cdot \frac{2}{3} = \frac{2}{300} = \frac{1}{150} \approx 0.00\overline{6}$

With the first four terms:

error $= \left| \frac{1}{1-2x} - (1+2x+4x^2+8x^3) \right| = 0.041\overline{6} > 0.00\overline{6}$, so continue

adding more terms until error $< 0.00\overline{6}$.

With the first $\boxed{7 \text{ terms}}$:

error $= \left| \frac{1}{1-2x} - (1+2x+4x^2+8x^3+16x^4+32x^5+64x^6) \right| = 0.00521 < 0.00\overline{6}$ ✓

1980 BC3:

a) $A = \sum_{n=1}^{\infty} \frac{4n}{n^2+1} \rightarrow b_n = \frac{4n}{n^2} = \frac{4}{n}$

Limit Comparison Test

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n}{n^2+1} \cdot \frac{n}{4} = \lim_{n\to\infty} \frac{4n^2}{4n^2+4} = 1 \rightarrow a_n$ & b_n same

$b_n = \frac{4}{n} = 4 \cdot \frac{1}{n} = 4(\text{Div.}) =$ Div. by p-series w/ $p=1$

Series A also $\boxed{\text{diverges by Limit Comp. Test}}$.

b) $S = \sum_{n=1}^{\infty} \frac{4n}{n^2+1} \cdot \frac{1}{2n} = \boxed{\sum_{n=1}^{\infty} \frac{4n}{2n^3+2n}} \rightarrow b_n = \frac{4n}{2n^3} = \frac{2}{n^2}$

c) Limit Comparison Test:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n}{2n^3+2n} \cdot \frac{n^2}{2} = \lim_{n\to\infty} \frac{4n^3}{4n^3+4n} = 1 \rightarrow$ a_n & b_n same

$b_n = \frac{2}{n^2} = 2 \cdot \frac{1}{n^2} = 2(\text{Con.}) =$ Converges by p-series w/ $p=2$

Series S also $\boxed{\text{converges by Limit Comp. Test}}$.

1981 BC3:

$S = \sum_{n=0}^{\infty} \left(\frac{t}{1+t}\right)^n$ where $t \neq 0$

a) $t=1$, so $S = \sum_{n=0}^{\infty} \left(\frac{1}{1+1}\right)^n = \left(\frac{1}{2}\right)^n$

Geometric Series w/ $r = 1/2$, and $a_0 = \left(\frac{1}{2}\right)^0 = 1$

$S = \frac{a_0}{1-r} = \frac{1}{1-1/2} = \frac{1}{1/2} = \boxed{2}$

b) $|r|<1$, so $\left|\frac{t}{1+t}\right| < 1 \rightarrow \frac{t}{1+t} < 1$ and $\frac{t}{1+t} > -1$

$t < 1+t$ $\qquad$ $t > -1-t$

$0 < 1$ $\qquad$ $2t > -1$

Always true $\qquad$ $\boxed{t > -1/2}$

c) $S = \frac{a_0}{1-r} > 10$

$a_0 = \left(\frac{t}{1+t}\right)^0 = 1$, $r = \frac{t}{1+t}$

$\frac{1}{1-\frac{t}{1+t}} > 10 \rightarrow \frac{1}{\frac{1+t}{1+t} - \frac{t}{1+t}} > 10 \rightarrow \frac{1}{\frac{1+\cancel{t}-\cancel{t}}{1+t}} > 10 \rightarrow \frac{1}{\frac{1}{1+t}} > 10$

$1+t > 10 \rightarrow \boxed{t > 9}$

1982 BC5 :

a) $f(x) = \ln(1+x)$ at $x=0$

$$\frac{1}{1+x} \rightarrow \begin{matrix} a_0 = 1 \\ r = -x \end{matrix} \rightarrow 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \ldots)\,dx$$

$$\ln(1+x) = \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n}}$$

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$

Ratio Test:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = \cancel{\lim_{n\to\infty} \frac{n}{n+1}}^{\,1} \cdot |x|^{\cancel{n}+1-\cancel{n}} = 1 \cdot |x|^1 = |x| = r$$

$|x| < 1 \rightarrow$ Cen: $x=0$, R: 1, IOC: $(-1, 1)$

$x = -1$: $(-1)^{n-1}(-1)^n \cdot \frac{1}{n} = \underbrace{(-1)^{2n-1}}_{\text{(2n-1 is always odd)}} \cdot \frac{1}{n} = \frac{-1}{n} \rightarrow$ Div. by p-series w/ $p=1$

$x = 1$: $(-1)^{n-1} \cdot \cancel{1^n} \cdot \frac{1}{n} = (-1)^{n-1} \cdot \frac{1}{n} \rightarrow$ Conv. Cond. by Alt. Series Test

IOC: $\boxed{(-1, 1]}$

c) $f(x) = \ln(1+x)$, so $\ln(3/2)$ is $f(1/2) = \ln(1 + 1/2)$

$$\text{error} = \left| \ln(3/2) - \left((1/2) - \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} \right) \right| = \boxed{0.00183}$$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{2n}$ vs. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$

Changes are that x has been replaced with x^2, and the function has been divided by 2.

Apply these changes to $\ln(1+x) \rightarrow \boxed{\frac{\ln(1+x^2)}{2}}$

1983 BC5:

$a_0 = 1, \quad a_1 = \frac{7}{n} \cdot 1 = \frac{7}{n} = \frac{7}{1} = 7, \quad a_2 = \frac{7}{n} \cdot 7 = \frac{49}{n} = \frac{49}{2}$

$a_3 = \frac{7}{n} \cdot \frac{49}{2} = \frac{243}{2n} = \frac{243}{2 \cdot 3} = \frac{243}{6}$

a) $\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots$

$1 + 7x + \frac{49}{2} x^2 + \frac{243}{6} x^3 = 1 + \frac{7x^1}{1!} + \frac{49x^2}{2!} + \frac{243x^3}{3!} + \ldots$

$$\boxed{1 + \frac{7^1 x^1}{1!} + \frac{7^2 x^2}{2!} + \frac{7^3 x^3}{3!} + \ldots + \frac{7^n x^n}{n!} = \frac{(7x)^n}{n!}}$$

b) $\sum_{n=0}^{\infty} \frac{(7x)^n}{n!}$

Ratio Test:

$\lim_{n\to\infty} \frac{|7x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|7x|^n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot |7x|^{\cancel{n}+1-\cancel{n}} = \lim_{n\to\infty} \frac{1}{n+1} \cdot |7x| = \frac{1}{\infty} \cdot |7x|$

$r = 0 \cdot |7x| = 0 < 1$ always, so converges for $\boxed{\text{all } x}$.

c) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(7x)^n}{n!} = 1 + \frac{7^1x^1}{1!} + \frac{7^2x^2}{2!} + \frac{7^3x^3}{3!} + \ldots + \frac{(7x)^n}{n!}$

$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

The given series is like e^x, except every x is replaced with $7x$.

$f(x) = e^{7x}$

$f'(x) = 7e^{7x}$

$f'(1) = 7e^{7(1)} = \boxed{7e^7}$

1984 BC4:

$$\sum_{n=1}^{\infty} \frac{x^n \cdot n^n}{3^n \cdot n!}$$

a) Ratio Test:

$$\lim_{n\to\infty} \frac{|x|^{n+1}(n+1)^{n+1}}{3^{n+1}\cdot(n+1)!} \cdot \frac{3^n \cdot n!}{|x|^n \cdot n^n} = \lim_{n\to\infty} \frac{n!\,(n+1)^{n+1}}{(n+1)!\, n^n} \cdot |x|^{\cancel{n}+1-\cancel{n}} \cdot 3^{\cancel{n}-\cancel{n}-1}$$

$$\lim_{n\to\infty} \frac{1}{\cancel{(n+1)}} \cdot \frac{\cancel{(n+1)} \cdot (n+1)^n}{n^n} \cdot |x|^1 \cdot 3^{-1} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \cdot \frac{|x|}{3} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot \frac{|x|}{3}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{\infty}\right)^\infty = (1+0)^\infty = 1^\infty \to \text{indeterminate form}$$

$$n \cdot \ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = \frac{\ln(1+0)}{0} = \frac{\ln 1}{0} = \frac{0}{0} = ? \to \text{L'Hopital's Rule}$$

$$\lim_{n\to\infty} \cancel{\frac{\frac{1}{1+1/n} \cdot \frac{-1}{n^2}}{\frac{-1}{n^2}}} = \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1+1/\infty} = \frac{1}{1+0} = \frac{1}{1} = 1$$

$$\cancel{\ln} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = 1 \to \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1 = e \to r = \frac{e|x|}{3}$$

$$\frac{e|x|}{3} < 1 \to e|x| < 3 \to |x| < \frac{3}{e} \to \boxed{\text{Radius} = \frac{3}{e}}$$

b)
$$\sum_{n=1}^{3} \frac{x^n \cdot n^n}{3^n \cdot n!} = \frac{x^1 \cdot \cancel{1^1}}{3^1 \cdot \cancel{1!}} + \frac{x^2 \cdot 2^2}{3^2 \cdot 2!} + \frac{x^3 \cdot \cancel{3^3}}{\cancel{3^3} \cdot 3!} = \frac{x}{3} + \frac{2}{9}x^2 + \frac{1}{6}x^3$$

$$f(-1) \approx \boxed{\frac{-1}{3} + \frac{2}{9} - \frac{1}{6}}$$

c) Error < next unused term, which is $n=4$.

$$n=4: \frac{x^4 \cdot 4^4}{3^4 \cdot 4!} \text{ at } x=-1: \frac{\cancel{(-1)^4}\, 4^4}{3^4 \cdot 4!} = \frac{4\cdot4\cdot4\cdot\cancel{4}}{3\cdot3\cdot3\cdot3\cdot1\cdot2\cdot3\cdot\cancel{4}} = \frac{64}{486} = \boxed{\frac{32}{243}}$$

1986 BC5:

a) $f(x)=\sqrt{1+x}=(1+x)^{1/2}$ $\quad f(0)=1$ $\quad 1+\dfrac{\frac{1}{2}x^1}{1!}-\dfrac{\frac{1}{4}x^2}{2!}+\dfrac{\frac{3}{8}x^3}{3!}$

$f'(x)=\frac{1}{2}(1+x)^{-1/2}$ $\quad f'(0)=\frac{1}{2}$

$f''(x)=\frac{-1}{4}(1+x)^{-3/2}$ $\quad f''(0)=\frac{-1}{4}$ $\quad \boxed{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3}$

$f'''(x)=\frac{3}{8}(1+x)^{-5/2}$ $\quad f'''(0)=\frac{3}{8}$

b) $g(x)=\sqrt{1+x^3}$, so replace each x with x^3.

$1+\frac{1}{2}x^3-\frac{1}{8}(x^3)^2+\frac{1}{16}(x^3)^3=\boxed{1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{1}{16}x^9}$

c) $h(x)=\int\left(1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{1}{16}x^9\right)dx = x+\frac{1}{8}x^4-\frac{1}{56}x^7+\frac{1}{160}x^{10}+4$

First 4 terms: $\boxed{4+x+\frac{1}{8}x^4-\frac{1}{56}x^7}$

1987 BC4:

a) $f(x)=\dfrac{1}{1-2x}\to \begin{matrix}a_0=1\\ r=2x\end{matrix}\to \boxed{1+2x+4x^2+8x^3+16x^4}+\dots+(2x)^n=\sum_{n=0}^{\infty}(2x)^n$

b) $\sum_{n=0}^{\infty}(2x)^n\to r=2x\to |2x|<1\to |x|<1/2\to$ Cen $x=0$, R: $1/2$, IOC: $(-1/2,1/2)$

$x=-1/2$: $(2\cdot-\frac{1}{2})^n=(-1)^n\to r=-1\to$ Div.

$x=1/2$: $(2\cdot\frac{1}{2})^n=(1)^n\to r=1\to$ Div.

IOC: $\boxed{(-1/2,1/2)}$

c) $\dfrac{1}{(1-2x)(1-x)}=\dfrac{A}{1-2x}+\dfrac{B}{1-x}\to A(1-x)+B(1-2x)=1$

$x=1$: $-B=1\to B=-1$

$x=\frac{1}{2}$: $\frac{1}{2}A=1\to A=2$

$\sum_{n=0}^{\infty}\dfrac{1}{(1-2x)(1-x)}=2\sum_{n=0}^{\infty}\dfrac{1}{1-2x}-\sum_{n=0}^{\infty}\dfrac{1}{1-x}\quad\left(\begin{matrix}a_0=1\\ r=x\end{matrix}\right)$

$2(1+2x+4x^2+8x^3+16x^4)-(1+x+x^2+x^3+x^4)$

$2+4x+8x^2+16x^3+32x^4-1-x-x^2-x^3-x^4=\boxed{1+3x+7x^2+15x^3+31x^4}$

1988 BC4:

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)} = \sum_{n=0}^{\infty} \frac{2^n x^n}{\ln(n+2)}$$

Ratio Test:

$$\lim_{n\to\infty} \frac{2^{n+1}|x|^{n+1}}{\ln(n+3)} \cdot \frac{\ln(n+2)}{2^n \cdot |x|^n} = \lim_{n\to\infty} \cancel{\frac{\ln(n+2)}{\ln(n+3)}}^{\,1} \cdot 2^{\cancel{n}+1-\cancel{n}} \cdot |x|^{\cancel{n}+1-\cancel{n}} = 2^1|x|^1 = 2|x| = r$$

$2|x| < 1 \to |x| < 1/2 \to$ Cen: $x=0$, R: $1/2$, IOC: $(-1/2, 1/2)$

$x = -1/2$: $\frac{2^n(-1/2)^n}{\ln(n+2)} = \frac{(2(-1/2))^n}{\ln(n+2)} = \frac{(-1)^n}{\ln(n+2)} \to$ Conv. Cond. by Alt. Series Test

$x = 1/2$: $\frac{2^n(1/2)^n}{\ln(n+2)} = \frac{(2(1/2))^n}{\ln(n+2)} = \frac{1}{\ln(n+2)} > \frac{1}{n} \to$ Div. by Direct Comp. Test

IOC: $\boxed{[-1/2, 1/2)}$

1990 BC5:

$f(x) = \frac{1}{x-1} = (x-1)^{-1}$ $\quad f(2) = 1$ $\quad 1 - \frac{1(x-2)^1}{1!} + \frac{2(x-2)^2}{2!} - \frac{6(x-2)^3}{3!}$

a) $f'(x) = -1(x-1)^{-2}$ $\quad f'(2) = -1$ $\quad \boxed{1-(x-2)+(x-2)^2-(x-2)^3+\dots+(-1)^n(x-2)^n}$

$f''(x) = 2(x-1)^{-3}$ $\quad f''(2) = 2$

$f'''(x) = -6(x-1)^{-4}$ $\quad f'''(2) = -6$

b) $\ln|x-1| = \int \frac{1}{x-1}dx = \int (1-(x-2)+(x-2)^2-(x-2)^3)dx$

$\ln|x-1| = x - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + C$

At $x=2$, $\ln|x-1| = \ln|2-1| = \ln 1 = 0$

$0 = 2 - 0 + 0 - 0 + C$, so $C = -2$

$\ln|x-1| = -2 + x - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \dots$

$\ln|x-1| = \boxed{(x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \dots + \frac{(-1)^{n+1}(x-2)^n}{n}}$

$n=1$ $\quad n=2$ $\quad n=3$ $\quad n=4$

c) $\ln 3/2 = \ln(x-1)$, so $\ln(5/2 - 1) = \ln(3/2) \rightarrow x = 5/2$.

$\ln 3/2 = 0.405$

We are looking for $0.355 < x < 0.455$

Add partial sums using the terms in part b until the sum is in this range

$2.5 - 2 = 0.5 \rightarrow$ No

$(2.5-2) - \frac{1}{2}(2.5-2)^2 = \boxed{0.375} \rightarrow$ Yes