Series Packet - Day 2: 1991 - 2002B

1991 BC5:

$f(t) = \frac{4}{1+t^2}$, $G(x) = \int_0^x f(t)\,dt$

a) $a_0 = 4$, $r = -t^2$ → $\boxed{4 - 4t^2 + 4t^4 - 4t^6 + \ldots + (-1)^n 4(t^2)^n = (-1)^n 4t^{2n}}$

b) $\int_0^x (4-4t^2+4t^4-4t^6)\,dt = \left(4t - \frac{4}{3}t^3 + \frac{4}{5}t^5 - \frac{4}{7}t^7\right)\Big|_0^x$

$$\boxed{4x - \frac{4}{3}x^3 + \frac{4}{5}x^5 - \frac{4}{7}x^7 + \ldots + \frac{(-1)^n 4x^{2n+1}}{2n+1}}$$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n 4x^{2n+1}}{2n+1}$

Ratio Test:

$$\lim_{n\to\infty} \frac{\cancel{4}|x|^{2n+3}}{2n+3} \cdot \frac{2n+1}{\cancel{4}|x|^{2n+1}} = \lim_{n\to\infty} \cancel{\frac{2n+1}{2n+3}}^{\,1} \cdot |x|^{\cancel{2n}+3-\cancel{2n}-1} = |x|^2 = r$$

$|x|^2 < 1$ → $|x| < 1$ → Cen: $x = 0$, R: 1, IOC: $(-1, 1)$

$x = -1$: $(-1)^n(-1)^{2n+1} \cdot \frac{4}{2n+1} = (-1)^{3n+1} \cdot \frac{4}{2n+1}$ → Conv. Cond. by Alt. Series Test

$x = 1$: $(-1)^n (1)^{\cancel{2n+1}} \cdot \frac{4}{2n+1} = (-1)^n \cdot \frac{4}{2n+1}$ → Conv. Cond. by Alt. Series Test

IOC: $\boxed{[-1, 1]}$

1992 BC6:

$\sum_{n=2}^{\infty} \frac{1}{n^p \cdot \ln(n)}$

a) $\ln 2, \ln 3, \ln 4, \ln 5, \ldots, \ln\infty$ are positive constant numbers

$\frac{1}{\ln(n)\cdot n^p} < \frac{1}{n^p}$ by Direct Comparison Test

$\frac{1}{n^p}$ converges by p-series when $p > 1$, so $\frac{1}{n^p \cdot \ln(n)}$ also $\boxed{\text{converges}}$.

b) For $p=1$, $\sum_{n=2}^{\infty} \frac{1}{n^p \cdot \ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n) \cdot n} = \frac{1}{\ln(n)} \cdot \frac{1}{n}$

$\frac{1}{n}$ diverges by p-series w/ $p=1$.

Constant (Diverges) = $\boxed{\text{Diverges}}$

c) $\sum_{n=2}^{\infty} \frac{1}{\ln(n) \cdot n^p} = \frac{1}{\ln(n)} \cdot \frac{1}{n^p}$

$\frac{1}{n^p}$ diverges when $p \le 1$ by p-series, which includes $0 \le p < 1$.

Constant (Diverges) = $\boxed{\text{Diverges}}$

1993 BC5:

$f(x) = e^{1/2x}$

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

$e^{1/2x} = 1 + \frac{1}{2}x + \frac{(1/2x)^2}{2!} + \frac{(1/2x)^3}{3!} + \dots + \frac{(1/2x)^n}{n!}$

$e^{1/2x} = \boxed{1 + \frac{x^1}{2^1 \cdot 1!} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \dots + \frac{x^n}{2^n \cdot n!}}$

b) $e^{\frac{1}{2}x} - 1 = \frac{x^1}{2^1 \cdot 1!} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \dots + \frac{x^n}{2^n \cdot n!}$

$\frac{e^{\frac{1}{2}x} - 1}{x} = \boxed{\frac{1}{2^1 \cdot 1!} + \frac{x^1}{2^2 \cdot 2!} + \frac{x^2}{2^3 \cdot 3!} + \dots + \frac{x^{n-1}}{2^n \cdot n!}}$

c) $g(x) = \frac{1}{2} + \frac{1}{8}x + \frac{1}{48}x^2 + \dots + \frac{x^{n-1}}{2^n \cdot n!}$

$g'(x) = \frac{1}{8} + \frac{1}{24}x + \dots + \frac{(n-1)x^{n-2}}{2^n \cdot n!}$

$g'(2) = \frac{1}{8} + \frac{1}{12} + \dots + \frac{(n-1) \cdot 2^{n-2}}{2^n \cdot n!} = \frac{(n-1)}{n!} \cdot 2^{\cancel{n}-2-\cancel{n}} = \frac{(n-1)}{2^2 \cdot n!} = \frac{(n-1)}{4n!}$

$g'(2) = \sum_{n=2}^{\infty} \frac{n-1}{4n!} = \sum_{n=1}^{\infty} \frac{n}{4(n+1)!}$ because both are $\frac{\text{term}}{4(\text{next})!}$

c) (continued)

$$g(x) = \frac{e^{1/2x} - 1}{x}$$

$$g'(x) = \frac{x \cdot \frac{1}{2} e^{1/2x} - (e^{1/2x} - 1)(1)}{x^2} = \frac{\frac{1}{2} x e^{1/2x} - e^{1/2x} + 1}{x^2}$$

$$g'(2) = \frac{\frac{1}{2} \cdot 2e^1 - e^1 + 1}{2^2} = \frac{\cancel{e} - \cancel{e} + 1}{4} = \frac{1}{4}$$

$$g'(2) = \boxed{\frac{1}{4} = \sum_{n=1}^{\infty} \frac{n}{4(n+1)!}}$$

1994 BC5:

$$f(x) = e^{-2x^2}$$

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$$e^{-2x^2} = 1 - 2x^2 + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \cdots + \frac{(-2x^2)^n}{n!}$$

$$e^{-2x^2} = \boxed{1 - \frac{2^1x^2}{1!} + \frac{2^2x^4}{2!} - \frac{2^3x^6}{3!} + \cdots + \frac{(-1)^n 2^n x^{2n}}{n!}}$$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n}}{n!} \rightarrow$ Ratio Test

$$\lim_{n\to\infty} \frac{2^{n+1}|x|^{2n+2}}{(n+1)!} \cdot \frac{n!}{2^n \cdot |x|^{2n}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot 2^{\cancel{n}+1-\cancel{n}} \cdot |x|^{\cancel{2n}+2-\cancel{2n}}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \cdot 2|x|^2 = \frac{1}{\infty} \cdot 2|x|^2 = 0 \cdot 2|x|^2 = 0 < 1 \text{ Always}$$

Converges for all $x \rightarrow$ IOC: $\boxed{(-\infty, \infty)}$

c) error $= \left| e^{-2x^2} - (1 - 2x^2 + 2x^4 - 4/3x^6) \right| = \boxed{0.00976}$ max error at $x = \pm 0.6$, which is still less than 0.02 as desired.

1996 BC2:

$$f(x) = 1 + \frac{x^1}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!}$$

a) Coefficient of $x^1 = \frac{f'(0)}{1!} = \frac{1}{2!} = \frac{1}{2} \rightarrow f'(0) = \frac{1}{2} \cdot 1! = \boxed{\frac{1}{2}}$

Coefficient of $x^{17} = \frac{f^{17}(0)}{17!} = \frac{1}{18!} \rightarrow f^{17}(0) = \frac{1}{18!} \cdot 17! = \frac{17!}{18!} = \boxed{\frac{1}{18}}$

b) $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \rightarrow$ Ratio Test

$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{|x|^n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!} \cdot |x|^{\cancel{n}+1-\cancel{n}} = \lim_{n\to\infty} \frac{1}{n+2} \cdot |x| = \frac{1}{\infty} \cdot |x|$

$= 0 \cdot |x| = 0 < 1$ Always $\rightarrow$ Converges for all $x \rightarrow$ IOC: $\boxed{(-\infty, \infty)}$

c) $g(x) = x\, f(x)$

$g(x) = \boxed{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n+1}}{(n+1)!}}$

d) $g(x) = \boxed{e^x - 1}$

$f(x) = \frac{g(x)}{x} = \frac{e^x - 1}{x}$, except undefined at $x = 0$

$f(0) = 1 + 0 + 0 + 0 + \cdots = 1 \rightarrow f(x) = \boxed{\begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}}$

1998

$f(0) = 5,\ f'(0) = -3,\ f''(0) = 1,\ f'''(0) = 4$

a) $5 - \frac{3x^1}{1!} + \frac{1x^2}{2!} + \frac{4x^3}{3!} = \boxed{5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3}$

$f(0.2) \approx 5 - 3(0.2) + \frac{1}{2}(0.2)^2 + \frac{2}{3}(0.2)^3 = \boxed{4.425}$

b) $g(x) = f(x^2)$

$5 - 3x^2 + \frac{1}{2}(x^2)^2 + \frac{2}{3}(x^2)^3 = \underbrace{5 - 3x^2 + \frac{1}{2}x^4}_{4^{th}\text{ degree}} + \frac{2}{3}x^6 \rightarrow \boxed{5 - 3x^2 + \frac{1}{2}x^4}$

c) $h(x) = \int_0^x f(t)\,dt = \int_0^x (5 - 3t + \frac{1}{2}t^2 + \frac{2}{3}t^3)\,dt$

$h(x) = (5t - \frac{3}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{6}t^4 + C)\Big|_0^x$

$h(x) = (5x - \frac{3}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \cancel{C}) - (0 - 0 + 0 + 0 + \cancel{C})$

$h(x) = \underbrace{5x - \frac{3}{2}x^2 + \frac{1}{6}x^3}_{3^{rd}\text{ degree}} + \frac{1}{6}x^4 \rightarrow \boxed{5x - \frac{3}{2}x^2 + \frac{1}{6}x^3}$

d) We know $f(0) = 5$ and $f(1) = 3$

$h(1) = \int_0^1 f(t)\,dt$

We do not know function f, or any other y values for f between $x=0$ and $x=1$. We cannot find the area, so we cannot integrate. We $\boxed{\text{cannot determine}}$ $h(1)$.

1999

$f(2) = -3,\ f'(2) = 5,\ f''(2) = 3,\ f'''(2) = -8$

a) $-3 + \frac{5(x-2)^1}{1!} + \frac{3(x-2)^2}{2!} - \frac{8(x-2)^3}{3!} = \boxed{-3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{4}{3}(x-2)^3}$

$f(1.5) \approx -3 + 5(1.5-2) + \frac{3}{2}(1.5-2)^2 - \frac{4}{3}(1.5-2)^3 = \boxed{-4.958}$

b) Lagrange error bound = next unused term = 4^{th} term

$\frac{f^4(x)(x-2)^4}{4!} = \frac{3(x-2)^4}{24} = \frac{1}{8}(x-2)^4$

Max error $= \frac{1}{8}(1.5-2)^4 = 0.0078125$

$-4.958 \pm 0.0078125 \rightarrow (-4.9658125, -4.9501875)$ is the range of possibilities for the exact value of $f(1.5)$.

Therefore, $\boxed{f(1.5) \neq -5}$, even with the maximum possible error.

c) $g(x) = f(x^2+2)$

$f(x) \approx -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{4}{3}(x-2)^3$ from part a

$g(x) = -3 + 5(x^2+\cancel{2}-\cancel{2}) + \frac{3}{2}(x^2+\cancel{2}-\cancel{2})^2 - \frac{4}{3}(x^2+\cancel{2}-\cancel{2})^3$

$g(x) = \underbrace{-3 + 5x^2 + \frac{3}{2}x^4}_{4^{th}\text{ degree}} - \frac{4}{3}x^6 \rightarrow \boxed{-3+5x^2+\frac{3}{2}x^4}$

$g'(0)=0 \quad g''(0)=10$

$-3+5x^2+\frac{3}{2}x^4 = -3+0x^1+5x^2+0x^3+\frac{3}{2}x^4 = -3+\frac{0x^1}{1!}+\frac{10x^2}{2!}+\frac{0x^3}{3!}+\frac{36x^4}{4!}$

$g'(0)=0$, so g has a critical point at $x=0$.

$g''(0)=10$, so g is concave up at $x=0$, so the critical point is a $\boxed{\text{minimum}}$ at $x=0$.

2001

$f(x) = \frac{1}{3^1} + \frac{2x}{3^2} + \frac{3x^2}{3^3} + \dots + \frac{(n+1)x^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{3^{n+1}}$

a) Ratio Test

$\lim_{n\to\infty} \frac{(n+2)|x|^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(n+1)\cdot|x|^n} = \lim_{n\to\infty} \cancel{\frac{n+2}{n+1}}^{1} \cdot |x|^{\cancel{n}+1-\cancel{n}} \cdot 3^{\cancel{n}+1-\cancel{n}-2}$

$r = 1\cdot|x|^1\cdot 3^{-1} = \frac{|x|}{3} < 1 \rightarrow |x| < 3 \rightarrow$ Cen: $x=0$, R: 3, IOC: $(-3,3)$

$x=-3$: $\frac{(n+1)(-3)^n}{3^{n+1}} = \frac{(-1)^n\cancel{3^n}(n+1)}{\cancel{3^n}\cdot 3} = (-1)^n \cdot \frac{n+1}{3} \rightarrow$ Div. bc $\lim_{n\to\infty} \frac{n+1}{3} \neq 0$ (n^{th} term test)

$x=3$: $\frac{(n+1)\cancel{(3)^n}}{\cancel{3^n}\cdot 3} = \frac{n+1}{3} \rightarrow$ Div. bc $\lim_{n\to\infty} \frac{n+1}{3} \neq 0$ (n^{th} term test) $\rightarrow$ IOC: $\boxed{(-3,3)}$

b) $\lim_{x\to 0} \frac{f(x)-1/3}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} =$ Deriv. of f at $x=0$

$f(x) = \frac{1}{3^1} + \frac{2x}{3^2} + \frac{3x^2}{3^3} + \frac{4x^3}{3^4} + \dots + \frac{(n+1)x^n}{3^{n+1}}$

$f'(x) = \frac{2}{3^2} + \frac{6x}{3^3} + \frac{12x^2}{3^4} + \dots + \frac{(n+1)n\,x^{n-1}}{3^{n+1}}$

$f'(0) = \frac{2}{3^2} + 0 + 0 + \dots = \frac{2}{3^2} = \boxed{\frac{2}{9}}$

c) $\int_0^1 \left(\frac{1}{3}+\frac{2}{3^2}x+\frac{3}{3^3}x^2\right)dx = \left(\frac{1}{3}x+\frac{1}{3^2}x^2+\frac{1}{3^3}x^3+\ldots+\frac{x^{n+1}}{3^{n+1}}\right)\Big|_0^1$

$\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\right)-\cancel{\left(0+0+0+\ldots\right)} = \boxed{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots+\frac{1}{3^{n+1}}}$

d) $a_0 = 1/3,\ r = 1/3 \rightarrow S = \frac{a_0}{1-r} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2} = \boxed{\frac{1}{2}}$

2002

$f(x) = \sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1} = \frac{2x}{1}+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{16x^4}{4}+\ldots+\frac{(2x)^{n+1}}{n+1}$

a) Ratio Test

$\lim_{n\to\infty}\frac{|2x|^{n+2}}{n+2}\cdot\frac{n+1}{|2x|^{n+1}} \rightarrow \lim_{n\to\infty}\cancel{\frac{n+1}{n+2}}^{1}\cdot|2x|^{\cancel{n}+2-\cancel{n}-1} = 1\cdot|2x|^1 = |2x| = r$

$|2x|<1 \rightarrow |x|<1/2 \rightarrow$ Cen: $x=0$, R: $1/2$, IOC: $(-1/2, 1/2)$

$x=-1/2$: $\frac{(2(-1/2))^{n+1}}{n+1} = (-1)^{n+1}\cdot\frac{1}{n+1} \rightarrow$ Conv. Cond. by Alt. Series Test

$x=1/2$: $\frac{(2(1/2))^{n+1}}{n+1} = \cancel{(1)^{n+1}}\cdot\frac{1}{n+1} \rightarrow$ Div. by Limit Comparison Test with $b_n=\frac{1}{n}$

IOC: $\boxed{[-1/2, 1/2)}$

b) $f'(x) = 2+4x+8x^2+16x^3+\ldots+\frac{\cancel{(n+1)}2^{n+1}x^n}{\cancel{(n+1)}} = 2^{n+1}x^n$

$f'(x) = \boxed{2+4x+8x^2+16x^3+\ldots+2^{n+1}x^n}$

c) $f'(-\frac{1}{3}) = 2+4(\frac{-1}{3})+8(\frac{-1}{3})^2+16(\frac{-1}{3})^3+\ldots+2^{n+1}(-\frac{1}{3})^n$

$f'(-\frac{1}{3}) = 2-\frac{4}{3}+\frac{8}{9}-\frac{16}{27}+\ldots$

$f'(-\frac{1}{3})$ is a geometric series with $a_0=2$ and $r=\frac{-2}{3}$

$S = \frac{a_0}{1-r} = \frac{2}{1--2/3} = \frac{2}{5/3} = 2\cdot\frac{3}{5} = \boxed{\frac{6}{5}}$

2002 (Form B)

$\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

a) $\ln\left(\frac{1}{1+3x}\right)$, so x is replaced with $-3x$: $\boxed{\sum_{n=1}^{\infty} \frac{(-3x)^n}{n} = \frac{(-1)^n(3x)^n}{n}}$

IOC for $\ln\left(\frac{1}{1-x}\right)$ is $-1 \le x < 1$.

IOC for $\ln\left(\frac{1}{1+3x}\right)$ is $\frac{-1}{-3} \le \frac{-3x}{-3} < \frac{1}{-3} \rightarrow \frac{1}{3} \ge x > \frac{-1}{3} \rightarrow \boxed{\left(\frac{-1}{3}, \frac{1}{3}\right]}$

b) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ is the series for $\ln\left(\frac{1}{1+3x}\right)$ in part a, with $x = 1/3$.

$\ln\left(\frac{1}{1+3(1/3)}\right) = \ln\left(\frac{1}{1+1}\right) = \boxed{\ln\left(\frac{1}{2}\right)}$

c) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges.

By p-series, $\frac{1}{n^{2p}}$ diverges when $2p \le 1 \rightarrow p \le \frac{1}{2}$

Test $\boxed{p = 1/2}$:

$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}} \rightarrow$ Conv. Cond. by Alternating Series Test ✓

$\sum_{n=1}^{\infty} \frac{1}{n^{2(1/2)}} = \frac{1}{n^1} = \frac{1}{n} \rightarrow$ Div. by Integral Test or p-series w/ $p=1$ ✓

d) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges.

$p \le 1$ $\qquad 2p > 1 \rightarrow p > \frac{1}{2}$ $\qquad$ Together: $\frac{1}{2} < p \le 1$

Test $\boxed{p=1}$:

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{n^1} \rightarrow$ Div. by p-series w/ $p=1$ ✓

$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} = \frac{1}{n^{2\cdot 1}} = \frac{1}{n^2} \rightarrow$ Conv. by p-series w/ $p=2$ ✓