Series Packet - Day 3: 2003 - 2007

2003

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots+\frac{(-1)^n x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$$

a) $\frac{0x^1}{1!} \to f'(0) = \boxed{0}$ and $\frac{-1}{3!}x^2 \to \frac{f''(0)}{2!} = \frac{-1}{3!} \to f''(0) = \frac{-2!}{3!} = \boxed{\frac{-1}{3}}$

At $x=0$, $f'(0)=0$, so there is a critical point. $f''(0) = \frac{-1}{3} < 0$, so the curve is concave down and the point is a $\boxed{\text{local min}}$.

b) error $\le$ next unused term $=$ 3rd term

error $\le \frac{x^4}{5!} = \frac{1^4}{5!} = \boxed{\frac{1}{120}} < \frac{1}{100}$, as desired.

c) $xy'+y = \cos x$

$$y = 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots+\frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$y' = \frac{-2x}{3!}+\frac{4x^3}{5!}-\frac{6x^5}{7!}+\cdots+\frac{(-1)^n(2n)x^{2n-1}}{(2n+1)!}$$

$$xy' = \frac{-2x^2}{3!}+\frac{4x^4}{5!}-\frac{6x^6}{7!}+\cdots+\frac{(-1)^n(2n)x^{2n}}{(2n+1)!}$$

$$xy'+y = 1-\frac{3x^2}{3!}+\frac{5x^4}{5!}-\frac{7x^6}{7!}+\cdots+\frac{(-1)^n(2n+1)x^{2n}}{(2n+1)!}$$

$$xy'+y = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!} = \boxed{\cos x}$$

2003 (Form B)

a) $f(2)=1$

$f'(2) = \frac{(1+1)!}{3^1} = \frac{2!}{3} = \frac{2}{3}$

$f''(2) = \frac{(2+1)!}{3^2} = \frac{3!}{9} = \frac{6}{9} = \frac{2}{3}$

$f'''(2) = \frac{(3+1)!}{3^3} = \frac{4!}{27} = \frac{24}{27} = \frac{8}{9}$

$$1+\frac{2/3(x-2)^1}{1!}+\frac{2/3(x-2)^2}{2!}+\frac{8/9(x-2)^3}{3!}$$

$$\boxed{1+\frac{2}{3}(x-2)+\frac{1}{3}(x-2)^2+\frac{4}{27}(x-2)^3}$$

General: $\dfrac{\frac{(n+1)!}{3^n}}{n!}\cdot(x-2)^n$

$$\frac{(n+1)!}{3^n\cdot n!}(x-2)^n = \boxed{\frac{n+1}{3^n}(x-2)^n}$$

b) $\sum_{n=0}^{\infty} \frac{(n+1)}{3^n}(x-2)^n \rightarrow$ Ratio Test

$$\lim_{n\to\infty} \frac{(n+2)|x-2|^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)|x-2|^n} = \lim_{n\to\infty} \overset{1}{\cancel{\frac{n+2}{n+1}}} \cdot |x-2|^{\cancel{n}+1-\cancel{n}} \cdot 3^{\cancel{n}-\cancel{n}-1}$$

$$r = 1\cdot|x-2|^{1}\cdot 3^{-1} = \frac{|x-2|}{3} < 1 \rightarrow |x-2|<3 \rightarrow \boxed{\text{Radius: } 3}$$

c) $f(x) = 1 + \frac{2}{3}(x-2) + \frac{1}{3}(x-2)^2 + \frac{4}{27}(x-2)^3$

$$g(x) = \int f(x)\,dx = x + \frac{1}{3}(x-2)^2 + \frac{1}{9}(x-2)^3 + \frac{1}{27}(x-2)^4 + C$$

$$2+0+0+0+C = 3 \rightarrow C = 1$$

$$g(x) = 1 + x + \frac{1}{3}(x-2)^2 + \frac{1}{9}(x-2)^3 + \frac{1}{27}(x-2)^4$$

want this term to match all the other $(x-2)$ terms

$$g(x) = \boxed{3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{9}(x-2)^3 + \ldots + \frac{(x-2)^n}{3^{n-1}}}$$

d) $$g(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n-1}} = \frac{(x-2)^n}{3^n \cdot 3^{-1}} = \frac{3(x-2)^n}{3^n} = 3\underbrace{\left(\frac{x-2}{3}\right)}_{\text{common ratio}}^n$$

$\left|\frac{x-2}{3}\right| < 1 \rightarrow |x-2|<3 \rightarrow$ Cen: $x=2$, R: $3 \rightarrow$ IOC: $(-1, 5)$

$x=-2$ is not in the interval of convergence, so $\boxed{\text{no}}$, the series for g does not converge at $x=-2$.

2004

$f(x) = \sin(5x + \pi/4)$ $\qquad$ $f(0) = \sqrt{2}/2$

a) $f'(x) = 5\cos(5x+\pi/4)$ $\qquad$ $f'(0) = 5\sqrt{2}/2$

$f''(x) = -25\sin(5x+\pi/4)$ $\qquad$ $f''(0) = -25\sqrt{2}/2$

$f'''(x) = -125\cos(5x+\pi/4)$ $\qquad$ $f'''(0) = -125\sqrt{2}/2$

$$\frac{\sqrt{2}}{2} + \frac{5/2\sqrt{2}\,x^1}{1!} - \frac{25/2\sqrt{2}\,x^2}{2!} - \frac{125/2\sqrt{2}\,x^3}{3!}$$

$$\boxed{\frac{\sqrt{2}}{2} + \frac{5}{2}\sqrt{2}\,x - \frac{25}{4}\sqrt{2}\,x^2 - \frac{125}{12}\sqrt{2}\,x^3}$$

b) Positive terms, $n = 0, 1, 4, 5, 8, 9, 12, 13, 16, 17, 20, 21$

Negative terms, $n = 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22$

General: $\dfrac{5^n\sqrt{2}}{2\cdot n!} \rightarrow n = 22: \boxed{\dfrac{-5^{22}\sqrt{2}}{2\cdot 22!}}$

c) error $\leq$ next unused term $=$ 4^{th} degree

$$\frac{5^4\sqrt{2}}{2\cdot 4!}x^4 = \frac{5^4\sqrt{2}}{2\cdot 4!}\left(\frac{1}{10}\right)^4 = 0.00184 < 0.01\text{, as desired}$$

d) $$G(x) = \int_0^x f(t)\,dt = \int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right)dt$$

$$G(x) = \left(\frac{\sqrt{2}}{2}t + \frac{5\sqrt{2}}{4}t^2 - \frac{25\sqrt{2}}{12}t^3\right)\Bigg|_0^x = \boxed{\frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3}$$

$G(0) = \int_0^0 f(t)\,dt = 0$, so $G(0) = 0$, so $C = 0$.

2004 (B)

$$T(x) = 7 - 9(x-2)^2 - 3(x-2)^3 = 7 - 0(x-2)^1 - 9(x-2)^2 - 3(x-2)^3$$

a) $f(2) =$ constant term $= \boxed{7}$

$f''(2)$: $-9(x-2)^2 \rightarrow \dfrac{f''(2)}{2!} = -9 \rightarrow f''(2) = \boxed{-18}$

b) $f'(2) = 0$ bc coefficient of linear term is 0.

$f'(2) = 0$ and $f''(2) = -18 < 0$

At $x = 2$, the curve has a critical point and is concave down, so the point is a $\boxed{\text{relative max}}$.

c) $f(0) \approx 7 - 9(-2)^2 - 3(-2)^3 = \boxed{-5}$

The approximation is centered at $x = 2$, so the coefficients do $\boxed{\text{not}}$ tell us anything about $f(0)$.

d) error ≤ next unused term = 4^{th} degree

$$\text{error} \le \frac{6(x-2)^4}{4!} = \frac{6(x-2)^4}{24} = \frac{1}{4}(x-2)^4$$

error at $x=0$ is $\le \frac{1}{4}(0-2)^4 = \frac{1}{4}\cdot 16 = 4$

$f(0) \approx -5$, so the true value is $-5 \pm 4 \to -9 < x < -1$.

Therefore, the actual value of $f(0)$ must be $\boxed{\text{negative}}$.

2005

$f(2) = 7,\ f'(2) = 0;\ f^3(2) = 0,\ f^5(2) = 0$

a) $f''(2) = \frac{(2-1)!}{3^2} = \frac{1!}{9} = \frac{1}{9}$

$f^4(2) = \frac{(4-1)!}{3^4} = \frac{3!}{81} = \frac{6}{81} = \frac{2}{27}$

$f^6(2) = \frac{(6-1)!}{3^6} = \frac{5!}{729} = \frac{120}{729} = \frac{40}{243}$

$$7 + \cancel{\frac{0(x-2)^1}{1!}} + \frac{1/9(x-2)^2}{2!} + \cancel{\frac{0(x-2)^3}{3!}} + \frac{2/27(x-2)^4}{4!} + \cancel{\frac{0(x-2)^5}{5!}} + \frac{40/243(x-2)^6}{6!}$$

$$\boxed{7 + \frac{1}{18}(x-2)^2 + \frac{1}{324}(x-2)^4 + \frac{1}{4374}(x-2)^6}$$

b) $(x-2)^{2n} \to$ even powers

$$\frac{\frac{(n-1)!}{3^n}}{n!} = \frac{(n-1)!}{3^n}\cdot\frac{1}{n!} = \frac{(n-1)!}{n!\cdot 3^n} = \frac{1}{n\cdot 3^n}$$

Make $\frac{1}{n\cdot 3^n}$ only evens: $\boxed{\frac{1}{(2n)\cdot 3^{2n}}}$

c) $\sum_{n=0}^{\infty} \frac{(x-2)^{2n}}{(2n)3^{2n}} \to$ Ratio Test

$$\lim_{n\to\infty} \frac{|x-2|^{2n+2}}{(2n+2)3^{2n+2}} \cdot \frac{(2n)3^{2n}}{|x-2|^{2n}} = \lim_{n\to\infty} \frac{\cancel{2n}}{\cancel{2n+2}}^{1} \cdot |x-2|^{\cancel{2n}+2-\cancel{2n}} \cdot 3^{\cancel{2n}-\cancel{2n}-2}$$

$r = 1\cdot|x-2|^2\cdot 3^{-2} = \frac{|x-2|^2}{9} < 1 \to |x-2|^2 < 9 \to |x-2| < 3 \to$

Cen: $x=2$

R: 3

IOC: $\boxed{(-1, 5)}$

$x=-1$: $\frac{(-3)^{2n}}{(2n)3^{2n}} = \frac{(-1)^{2n}\cdot\cancel{3^{2n}}}{(2n)\cdot\cancel{3^{2n}}} = (-1)^{2n}\cdot\frac{1}{2n} \to 2n$ is always even, so always positive

Div. by Limit Comp. Test w/ $b_n = \frac{1}{n}$

$x=5$: $\frac{\cancel{3^{2n}}}{(2n)\cancel{3^{2n}}} = \frac{1}{2n} \to$ Div. w/ Limit Comp. Test w/ $b_n = \frac{1}{n}$

2005 (B)

a) $f'(0) = 0$ bc horizontal tangent

$$f''(0) = \frac{(-1)^{2+1}(2+1)!}{5^2(2-1)^2} = \frac{(-1)^3 \cdot 3!}{25 \cdot \cancel{1^2}} = \frac{-6}{25} < 0$$

At $x=0$ there is a critical point and f is concave down, so there is a $\boxed{\text{relative max}}$ at $x=0$.

b) $$f'''(0) = \frac{\cancel{(-1)^4}\, 4!}{5^3 \cdot 2^2} = \frac{24}{125 \cdot 4} = \frac{6}{125}$$

$$6 + \frac{0x^1}{1!} - \frac{6/25\, x^2}{2!} + \frac{6/125\, x^3}{3!} = 6 - \frac{6}{50}x^2 + \frac{6}{750}x^3 = \boxed{6 - \frac{3}{25}x^2 + \frac{1}{125}x^3}$$

c) General term: $$\frac{\frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2}}{n!} \cdot x^n = \frac{(-1)^{n+1}(n+1)!\, x^n}{5^n(n-1)^2 n!} = \frac{(-1)^{n+1}(n+1)x^n}{5^n(n-1)^2}$$

Ratio Test:

$$\lim_{n\to\infty} \frac{(n+2)|x|^{n+1}}{5^{n+1} \cdot n^2} \cdot \frac{5^n(n-1)^2}{(n+1)|x|^n} = \lim_{n\to\infty} \frac{(n+2)\overset{1}{\cancel{(n-1)^2}}}{\cancel{n^2}(n+1)} \cdot 5^{\cancel{n}-\cancel{n}-1} \cdot |x|^{\cancel{n}+1-\cancel{n}}$$

$$r = 1 \cdot 5^{-1} \cdot |x|^1 = \frac{|x|}{5} < 1 \rightarrow |x| < 5 \rightarrow \text{Center: } x = 0$$

$\boxed{\text{Radius: } 5}$

2006

a) $$\sum_{n=1}^{\infty} \frac{(-1)^n n x^n}{n+1} \rightarrow \text{Ratio Test}$$

$$\lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{n+2} \cdot \frac{n+1}{n|x|^n} = \lim_{n\to\infty} \frac{\overset{1}{\cancel{n^2+2n+1}}}{\cancel{n^2+2n}} \cdot |x|^{\cancel{n}+1-\cancel{n}} = 1 \cdot |x|^1 = |x| = r$$

$|x| < 1 \rightarrow$ Cen: $x = 0$, R: 1 $\rightarrow$ IOC: $\boxed{(-1, 1)}$

$x = -1$: $\frac{(-1)^n(-1)^n n}{n+1} = \cancel{(1)^n} \cdot \frac{n}{n+1} \rightarrow$ Div bc $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

$x = 1$: $\frac{(-1)^n \cancel{(1)^n}\, n}{n+1} = (-1)^n \cdot \frac{n}{n+1} \rightarrow$ Div bc $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

b) $f(x) = \cancel{\frac{-1}{2}x} + \frac{2}{3}x^2 - \frac{3}{4}x^3 + \dots + \frac{(-1)^n n \cdot x^n}{n+1}$

$g(x) = 1 - \cancel{\frac{1}{2}x} + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \dots + \frac{(-1)^n x^n}{(2n)!}$

$y = f(x) - g(x) = -1 + \frac{5}{8}x^2 - \frac{539}{720}x^3 + \dots$

$y' = \frac{5}{4}x - \frac{539}{240}x^2 \rightarrow y'(0) = 0 - 0 = 0 \rightarrow$ critical point at $x=0$

$y'' = \frac{5}{4} - \frac{539}{120}x \rightarrow y''(0) = \frac{5}{4} - 0 = \frac{5}{4} > 0 \rightarrow$ concave up at $x=0$

At $x=0$, y has a $\boxed{\text{relative min}}$.

2007

$f(x) = e^{-x^2}$

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

$e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots + \frac{(-x^2)^n}{n!}$

$e^{-x^2} = \boxed{1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!}}$

b) $1 - x^2 - f(x) = \cancel{1} - \cancel{x^2} - \cancel{1} + \cancel{x^2} - \frac{x^4}{2!} + \frac{x^6}{3!} - \frac{x^8}{4!} + \frac{x^{10}}{5!} \dots$

$\frac{1 - x^2 - f(x)}{x^4} = \frac{-1}{2!} + \frac{x^2}{3!} - \frac{x^4}{4!} + \frac{x^6}{5!}$

$\lim_{x \to 0} \frac{1 - x^2 - f(x)}{x^4} = \frac{-1}{2!} + 0 - 0 + 0 = \frac{-1}{2!} = \boxed{\frac{-1}{2}}$

c) $\int_0^x e^{-t^2}dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!}\right)dt = \left(t - \frac{1}{3}t^3 + \frac{1}{10}t^5 - \frac{1}{42}t^7\right)\Big|_0^x = \boxed{x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7}$

$\int_0^{1/2} e^{-t^2}dt$ is when $x = \frac{1}{2}$: $\left(\frac{1}{2}\right) - \frac{1}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{2} - \frac{1}{24} = \boxed{\frac{11}{24}}$

(with first two terms)

d) error $\leq$ next unused term $= 5^{\text{th}}$ degree

error $\leq \frac{1}{10}x^5 = \frac{1}{10}\left(\frac{1}{2}\right)^5 = \frac{1}{10} \cdot \frac{1}{32} = \boxed{\frac{1}{320}} < \frac{1}{200}$, as desired.