Series Packet - Day 4: 2007B - 2010

2007B

$f(x) = 6e^{-\frac{1}{3}x}$

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$e^{-\frac{1}{3}x} = 1 - \frac{1}{3}x + \frac{(-\frac{1}{3}x)^2}{2!} + \frac{(-\frac{1}{3}x)^3}{3!} + \cdots + \frac{(-\frac{1}{3}x)^n}{n!}$

$e^{-\frac{1}{3}x} = 1 - \frac{x}{3^1 \cdot 1!} + \frac{x^2}{3^2 \cdot 2!} - \frac{x^3}{3^3 \cdot 3!} + \cdots + \frac{(-1)^n x^n}{3^n \cdot n!}$

$6e^{-\frac{1}{3}x} = \boxed{6 - \frac{6x}{3^1 \cdot 1!} + \frac{6x^2}{3^2 \cdot 2!} - \frac{6x^3}{3^3 \cdot 3!} + \cdots + \frac{(-1)^n 6x^n}{3^n \cdot n!}}$

b) $g(x) = \int_0^x f(t)\,dt = \int_0^x \left(6 - \frac{6t}{3^1 \cdot 1!} + \frac{6t^2}{3^2 \cdot 2!} - \frac{6t^3}{3^3 \cdot 3!}\right)dt = \left(6t - \frac{6t^2}{2 \cdot 3^1 \cdot 1!} + \frac{6t^3}{3 \cdot 3^2 \cdot 2!} - \frac{6t^4}{4 \cdot 3^3 \cdot 3!}\right)\Big|_0^x$

$g(x) = \boxed{6x - \frac{6x^2}{2 \cdot 3^1 \cdot 1!} + \frac{6x^3}{3 \cdot 3^2 \cdot 2!} - \frac{6x^4}{4 \cdot 3^3 \cdot 3!} + \cdots + \frac{(-1)^n 6x^{n+1}}{3^n \cdot (n+1)n!} = \frac{(-1)^n 6x^{n+1}}{3^n (n+1)!}}$

c) $f(x) = 6 - 2x + \frac{1}{3}x^2 - \frac{1}{27}x^3 + \cdots + \frac{(-1)^n 6x^n}{3^n \cdot n!}$

$f'(x) = -2 + \frac{2}{3}x - \frac{1}{9}x^2 + \cdots + \frac{(-1)^n 6nx^{n-1}}{3^n \cdot n!} = \frac{(-1)^n 6x^{n-1}}{3^n (n-1)!}$

$f'(ax) = -2 + \frac{2}{3}ax - \frac{1}{9}(ax)^2 + \cdots + \frac{(-1)^n 6(ax)^{n-1}}{3^n (n-1)!}$

$f'(ax) = -2 + \frac{2}{3}ax - \frac{1}{9}a^2x^2 + \cdots + \frac{(-1)^n 6a^{n-1}x^{n-1}}{3^n (n-1)!}$

$kf'(ax) = -2k + \frac{2}{3}akx - \frac{1}{9}ka^2x^2 + \cdots + \frac{(-1)^n 6ka^{n-1}x^{n-1}}{3^n (n-1)!}$

$\underbrace{-2k} + \frac{2}{3}akx - \frac{1}{9}ka^2x^2 = \underbrace{1} + x + \frac{x^2}{2!} \rightarrow -2k = 1 \rightarrow \boxed{k = -\frac{1}{2}}$

$1 - \frac{1}{3}ax + \frac{1}{18}a^2x^2 = 1 + 1x + \frac{x^2}{2!} \rightarrow -\frac{1}{3}a = 1 \rightarrow \boxed{a = -3}$

2008

a) $h(2) + \frac{h'(2)(x-2)^1}{1!} = \boxed{80 + 128(x-2)}$

$h(1.9) \approx 80 + 128(1.9-2) = \boxed{67.2}$

$h''(2) = \frac{488}{3} > 0$, so h is concave up at $x=2$. Therefore, the linear approximation of $h(1.9)$ is $\boxed{\text{less}}$ than the true value.

b) $80 + 128(x-2) + \frac{h''(2)(x-2)^2}{2!} + \frac{h'''(2)(x-2)^3}{3!}$

$80 + 128(x-2) + \frac{488/3\,(x-2)^2}{2} + \frac{448/3\,(x-2)^3}{6}$

$$\boxed{80 + 128(x-2) + \frac{244}{3}(x-2)^2 + \frac{224}{9}(x-2)^3}$$

$h(1.9) \approx 80 + 128(-0.1) + \frac{244}{3}(-0.1)^2 + \frac{224}{9}(-0.1)^3 = \boxed{67.988}$

c) error $\leq$ next unused term = 4^{th} order

error $\leq \frac{h^4(2)(x-2)^4}{4!} = \frac{584/9\,(1.9-2)^4}{4!} = \boxed{2.704 \times 10^{-4}} < 3 \times 10^{-4}$, as desired.

2008 (Form B)

$f(x) = \frac{2x}{1+x^2}$

a) $a_0 = 2x$, $r = -x^2 \rightarrow \boxed{2x - 2x^3 + 2x^5 - 2x^7 + \ldots + (-1)^n 2x^{2n+1}}$

b) $2(1) - 2(1)^3 + 2(1)^5 - 2(1)^7 + \ldots$

Partial sums alternate : 2, 0, 2, 0, ... (Diverges)

The series does $\boxed{\text{not converge}}$ in general, so it cannot converge to $f(1)$.

c) $\ln(1+x^2) = \int \frac{2x}{1+x^2}\,dx = \int (2x - 2x^3 + 2x^5 - 2x^7)\,dx$

$\ln(1+x^2) = \boxed{x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8}$

d) Within 0.01 of $\ln(5/4) = \ln(1+x^2)$, so $x^2 = 1/4 \rightarrow x = 1/2$

$\ln(5/4) = 0.223 \pm 0.01 \rightarrow 0.213 < x < 0.233$

Add terms one at a time as partial sums until we get there.

$x^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25 \rightarrow$ No

$\left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 = \frac{1}{4} - \frac{1}{32} = \boxed{0.21875} \rightarrow$ Yes

2009

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

$$e^{(x-1)^2} = \boxed{1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \dots + \frac{(x-1)^{2n}}{n!}}$$

b) $e^{(x-1)^2} - 1 = (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \frac{(x-1)^8}{4!}$

$$\frac{e^{(x-1)^2} - 1}{(x-1)^2} = \boxed{1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{3!} + \frac{(x-1)^6}{4!} + \dots + \frac{(x-1)^{2n-2}}{n!}}$$

c) $\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{n!} \rightarrow$ Ratio Test

$$\lim_{n\to\infty} \frac{|x-1|^{2n}}{(n+1)!} \cdot \frac{n!}{|x-1|^{2n-2}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot |x-1|^{\cancel{2n}-\cancel{2n}+2} = \lim_{n\to\infty} \frac{1}{n+1} \cdot |x-1|^2$$

$$r = \frac{1}{\infty+1} \cdot |x-1|^2 = 0 \cdot |x-1|^2 = 0 < 1 \text{ Always} \rightarrow \text{IOC: } \boxed{(-\infty, \infty)}$$

d) $f(x) = 1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{3!} + \frac{(x-1)^6}{4!} + \dots + \frac{(x-1)^{2n-2}}{n!}$

$$f'(x) = (x-1) + \frac{2}{3}(x-1)^3 + \frac{1}{4}(x-1)^5 + \dots + \frac{(2n-2)(x-1)^{2n-3}}{n!}$$

$$f''(x) = \underset{\substack{\uparrow \\ +}}{1} + \underbrace{2(x-1)^2}_{+} + \underbrace{\frac{5}{4}(x-1)^4}_{+} + \dots + \frac{(2n-3)(2n-2)(x-1)^{2n-4}}{n!}$$

In $f''(x)$, all exponents are even and all coefficients are positive, so all of the terms in $f''(x)$ are positive. No matter how many terms we add, the sum will be positive and $f''(x)$ stays positive. Since $f''(x)$ never changes signs, f has $\boxed{\text{no POI's}}$.

2009 (Form B)

$$f(x) = 1 + (x+1) + (x+1)^2 + \dots + (x+1)^n = \sum_{n=0}^{\infty} (x+1)^n$$

a) $r = x+1 \rightarrow |x+1| < 1 \rightarrow$ Cen: $x=-1$, R: $1 \rightarrow$ IOC: $\boxed{(-2,0)}$

$x=-2$: $(-2+1)^n = (-1)^n \rightarrow r=-1 \rightarrow$ Div. by Geo. Series

$x=0$: $(0+1)^n = (1)^n \rightarrow r=1 \rightarrow$ Div. by Geo. Series

b) $$S = \frac{a_0}{1-r} = \frac{1}{1-(x+1)} = \frac{1}{\cancel{1}-x-\cancel{1}} = \frac{1}{-x} = \boxed{-\frac{1}{x}}$$

c) $$g(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} \underbrace{\frac{-1}{t}}_{\text{from part b)}}\,dt = -\ln|t|\Big|_{-1}^{x} = -\ln|x| + \overset{0}{\cancel{\ln|-1|}} = -\ln|x|$$

$$g\left(-\tfrac{1}{2}\right) = -\ln\left|-\tfrac{1}{2}\right| = \boxed{-\ln(1/2)}$$

d) $h(x) = f(x^2-1)$

$$h(x) = 1 + (x^2-\cancel{1}+\cancel{1}) + (x^2-\cancel{1}+\cancel{1})^2 + \dots + (x^2-\cancel{1}+\cancel{1})^n$$

$$h(x) = \boxed{1 + x^2 + x^4 + \dots + x^{2n}}$$

$$h\left(\tfrac{1}{2}\right) = 1 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^4 + \dots = 1 + \frac{1}{4} + \frac{1}{16} + \dots \qquad a_0 = 1,\ r = 1/4$$

$$S = \frac{a_0}{1-r} = \frac{1}{1-1/4} = \frac{1}{3/4} = \boxed{\frac{4}{3}}$$

2010

a) $\cos x = \boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}}$

$\cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^n x^{2n}}{(2n)!}$

$\frac{\cos x - 1}{x^2} = \boxed{\frac{-1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots \frac{(-1)^n x^{2n-2}}{(2n)!}}$

b) $\frac{f'(0)x^1}{1!} = 0x^1 \rightarrow f'(0) = 0 \rightarrow$ Critical point at $x=0$ $\boxed{\text{Rel. Min.}}$

$\frac{f''(0)x^2}{2!} = \frac{1}{24}x^2 \rightarrow f''(0) = \frac{2!}{24} = \frac{2}{24} = \frac{1}{12} > 0 \rightarrow$ Concave up at $x=0$

c) $g(x) = 1 + \int_0^x f(t)\,dt = 1 + \int_0^x \left(\frac{-1}{2!} + \frac{t^2}{4!} - \frac{t^4}{6!} + \dots\right)dt$

$g(x) = \boxed{1 - \frac{t}{2!} + \frac{t^3}{3 \cdot 4!} - \frac{t^5}{5 \cdot 6!}}$

d) $g(1) \approx 1 - \frac{1}{2} + \frac{1}{72} = \frac{72}{72} - \frac{36}{72} + \frac{1}{72} = \boxed{\frac{37}{72}}$

error $\leq$ next unused term = 5^{th} order

error $\leq \frac{t^5}{5 \cdot 6!} = \frac{1^5}{5 \cdot 6!} = \boxed{\frac{1}{5 \cdot 6!}} < \frac{1}{6!}$, as desired.